7.2. SIMPLEST LINEAR MODEL OF THERMOCLINE

The temperature (and likewise salinity and density) field in the ocean has one clearly pronounced feature: practically all temperature changes (in vertical as well as horizontal directions) are concentrated in the upper kilometre layer which is usually referred to as main thermocline or simply as thermocline (Fig. 7.2). The bottom thickness of the water of the ocean (below the thermocline) has almost constant temperature which, in essence, does not depend on the thermal conditions at the surface of the ocean. Such a pattern is true for the entire world ocean, except, may be, in high latitudes. Starting from general reasoning, it may be assumed that the thermocline is nothing else but a specific thermal boundary layer of the ocean. It is the task of theory to explain, first of all, the parameters on which the characteristic thickness of this boundary layer depends. Besides, the existence of a thermal boundary layer in the open ocean raises naturally a number of new problems also for the theory of coastal boundary currents. It has already been shown in the last section how complicated the structure of coastal boundary layers becomes (even in the case of a homogeneous ocean) when one steps over from two-dimensional to three-dimensional models.

A start will be made with the simplest possible model. Assume that there is no wind, i.e., the motion is due to purely thermal causes. However, then one need not take into consideration vertical turbulent exchange (at least, outside the bottom Ekman boundary layer). Disregard also non-linear inertial terms and horizontal turbulent transfer; for the sake of simplicity, restrict consideration to the β-plane approximation, so that

$$-fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}, \tag{7.2.1}$$

$$fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}, \tag{7.2.2}$$

$$\frac{\partial p}{\partial z} = g\rho, \tag{7.2.3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{7.2.4}$$

where, as usually, the x-axis is directed to the east, the y-axis to the north, the z-axis downwards, ρ_0 is the mean density in the ocean, $f = f_0 + \beta(y - y_0)$ and the remaining notation is as before.

It will be assumed that the density depends only on the temperature, and besides linearly, so that

$$\rho = \rho_0[1 - \alpha(T - T_0)], \tag{7.2.5}$$

where T_0 is the temperature averaged over the entire ocean and α the constant coefficient of thermal expansion.

The equation of heat transfer is non-linear, and this fact is the cause of basic difficulties encountered in the construction of a theory. As a first step, linearization will be introduced (the non-linear theory will be considered in the next section). Thus, let the temperature field in the ocean assume the form

$$T = T_s - Gz + T'(x, y, z) , \qquad G > 0 , \tag{7.2.6}$$

where $T = T_s - Gz$(T_s, G = constant) is a certain mean distribution of temperature in the ocean and T' is a perturbation. Assuming the perturbation of the temperature field to be small, linearize the equation of heat transfer (4.5.6). In addition, assume the coefficients of vertical and horizontal heat conductivity (K_H, K_L) to be constant, to obtain, finally,

$$-Gw = K_H \frac{\partial^2 T'}{\partial z^2} + K_L \Delta_h T' . \tag{7.2.7}$$

The boundary conditions will be formulated next. Thermal processes at the ocean surface are extremely complicated, and their discussion is not required with the problem under consideration. As regards a study of the effect of the formation of a thermal boundary layer, it is sufficiently simple to specify the temperature at the ocean surface. Recalling likewise the kinematic boundary condition, one has

$$T' = \theta(y) , \qquad w = 0 \qquad \text{for } z = 0 , \tag{7.2.8}$$

where $\theta(y)$ is a known function (the temperature at the surface of the ocean changes chiefly in a meridional direction).

Let it be assumed that there is no perturbation of the temperature at the ocean floor for $z = H$. Furthermore, since vertical turbulent exchange has been neglected in the equations of motion, one may impose at the bottom only a condition of no flow. Thus,

$$T' = 0 , \qquad w = 0 \qquad \text{for } z = H . \tag{7.2.9}$$

A formulation of boundary conditions at shores which are assumed to be sheer cliffs is very specific. In general, there is no heat flux at a shore (for example, for $x = 0$ and $x = L$) and the horizontal velocity must also vanish. However, since horizontal turbulent exchange is absent from the equations of motion as well as are non-linear terms, one is forced to forget about fulfillment of all conditions at the shore. Clearly, it is impossible to violate the no flow condition: The total mass of fluid in the basin must remain constant. However, if, for example, one has $u = 0$ for $x = 0$, then, by (7.2.2), $\partial p/\partial y$ = 0, which means, by (7.2.3) and (7.2.5), that also $\partial T'/\partial y = 0$ for $x = 0$. Therefore it will be simplest to write

$$T' = 0 \qquad \text{for } x = 0 , \qquad T' = 0 \qquad \text{for } x = L . \tag{7.2.10}$$

Thus, the thermal and dynamic boundary conditions could be consolidated in the single condition (7.2.10). The conditions at the zonal boundaries of the region will not be considered here.

It will be convenient to reduce the problem to a single equation for T

Eliminating from (7.2.1) and (7.2.2) the pressure p and employing (7.2.4), one obtains

$$\beta v = f\frac{\partial w}{\partial z}\,. \tag{7.2.11}$$

Differentiating this equation with respect to z and taking (7.2.3) and (7.2.5) into account, one finds

$$\frac{\partial^2 w}{\partial z^2} = -\frac{g\alpha\beta}{f^2}\frac{\partial T'}{\partial x}\,.$$

Finally, substituting into this equation, in accordance with (7.2.7), the expression for w in terms of T', one obtains

$$K_H\frac{\partial^4 T'}{\partial z^4} + K_L\Delta_h\left(\frac{\partial^2 T'}{\partial z^2}\right) - \frac{g\alpha\beta G}{f^2}\frac{\partial T'}{\partial x} = 0\,. \tag{7.2.12}$$

Conditions (7.2.8), (7.2.9) and (7.2.10) will now be rewritten so that they only involve the temperature perturbation T':

$$T' = \theta(y)\,,\qquad K_H\frac{\partial^2 T'}{\partial z^2} + K_L\Delta_h T' = 0 \qquad \text{for } z = 0\,,$$

$$T' = 0\,,\ K_H\frac{\partial^2 T'}{\partial z^2} + K_L\Delta_h T' = 0 \qquad \text{for } z = H\,,$$

$$T' = 0 \qquad \text{for } x = 0\,,\qquad T' = 0 \qquad \text{for } x = L\,. \tag{7.2.13}$$

Thus, the study of the thermal boundary layer has been reduced to analysis of the problem (7.2.12) and (7.2.13). These equations will now be written in non-dimensional form.

Choose L and H as characteristic scales in horizontal and vertical directions, respectively. Let θ_0 be the characteristic value of the function $\theta(y)$; it will be quite natural to adopt θ_0 as characteristic scale for T'. Writing (7.2.12) and (7.2.13) in terms of non-dimensional quantities (denoted below by the former symbols), one obtains

$$\epsilon^4\frac{\partial^4 T'}{\partial z^4} + \gamma\epsilon^4\Delta_h\left(\frac{\partial^2 T'}{\partial z^2}\right) - \frac{1}{f^2(y)}\frac{\partial T'}{\partial x} = 0\,, \tag{7.2.14}$$

$$T' = \theta(y)\,,\ \frac{\partial^2 T'}{\partial z^2} + \gamma\Delta_h T' = 0 \qquad \text{for } z = 0\,,$$

$$T' = 0\,,\qquad \frac{\partial^2 T'}{\partial z^2} + \gamma\Delta_h T' = 0 \qquad \text{for } z = 1\,,$$

$$T' = 0 \qquad \text{for } x = 0\,,\qquad T' = 0 \qquad \text{for } x = 1\,, \tag{7.2.15}$$

where

$$\epsilon = \frac{H_0}{H}, \qquad H_0 = \left(\frac{K_H f_0^2 L}{g\alpha\beta G}\right)^{1/4}, \qquad \gamma = \frac{K_L}{K_H}\left(\frac{H}{L}\right)^2. \tag{7.2.16}$$

If $K_H = 1$ cm^2/sec, $K_L = 10^7$ cm^2/sec, $H = 4$ km, $L = 5000$ km, $\alpha = 2.5 \cdot 10^{-4}$ $(^{0}\mathrm{C})^{-1}$, $G = 10^{-4}$ $(^{0}\mathrm{C})$/cm, then $\gamma \simeq 6$, $H_0 \simeq 0.3$ km, $\epsilon \simeq 0.08$. These estimates are very approximate; however, it will be assumed in the sequel that the parameter γ is finite and the parameter ϵ small. The asymptotic of the solution of problem (7.2.14) and (7.2.15) for small ϵ yields completely satisfactory understanding of the peculiarities of the solution of the problem also for not very small ϵ.

It is already clear, starting from these considerations, that the internal characteristic scale H_0 gives the order of magnitude of the thickness of the thermal boundary layer, or of the thermocline, in the ocean. Therefore it may be assumed that qualitatively the model under consideration actually describes the effect of formation of the thermal boundary layer in the ocean. The structure of the boundary layers will now be studied in greater detail.

The temperature perturbation T' outside the boundary layers vanishes according to (7.2.14) and (7.2.15) (internal solution). Obviously, the non-dimensional thickness of the thermal boundary layer is of order ϵ. It is not difficult to determine also the order of the non-dimensional thickness of the coastal boundary layer $O(\epsilon^2)$ from the condition that the terms $\gamma\epsilon^4 \Delta_h(\partial^2 T'/\partial z^2)$ and $\partial T'/\partial x$ must be of equal order of magnitude. Thus, the solution of Problems (7.2.14) and (7.2.15) outside the boundary layers at zonal boundaries will be sought in the form

$$T' = T_s(x, y, \xi) + \ldots + T_W(\zeta, y, \xi) + \ldots + T_E(\eta, y, \xi) + \ldots, \tag{7.2.17}$$

where $\xi = z/\epsilon$, $\zeta = x/\epsilon^2$, $\eta = (1-x)/\epsilon^2$ and all functions T_s, T_W, T_E must decay exponentially for large ξ, ζ and η.

Substitution of (7.2.17) into (7.2.14) and (7.2.15) leads to the relations

$$\frac{\partial^4 T_s}{\partial \xi^4} - \frac{1}{f^2}\frac{\partial T_s}{\partial x} = 0, \qquad 0 < x < 1, \qquad \xi > 0, \tag{7.2.18}$$

$$T_s = \theta(y), \qquad \frac{\partial^2 T_s}{\partial \xi^2} = 0 \qquad \text{for } \xi = 0, \tag{7.2.19}$$

$$\gamma\frac{\partial^4 T_W}{\partial \xi^2 \partial \zeta^2} - \frac{1}{f^2}\frac{\partial T_W}{\partial \zeta} = 0, \qquad \xi, \zeta > 0, \tag{7.2.20}$$

$$T_s + T_W = 0 \qquad \text{for } \zeta = 0, \tag{7.2.21}$$

$$T_W = 0 \qquad \text{for } \xi = 0, \tag{7.2.22}$$

$$\gamma\frac{\partial^4 T_E}{\partial \xi^2 \partial \eta^2} + \frac{1}{f^2}\frac{\partial T_E}{\partial \eta} = 0, \qquad \xi, \eta > 0, \tag{7.2.20'}$$

$$T_s + T_E = 0 \qquad \text{for } \eta = 0\,, \tag{7.2.21'}$$

$$T_E = 0 \qquad \text{for } \xi = 0\,. \tag{7.2.22'}$$

A start will be made with problems (7.2.18) and (7.2.19). Introduce the Fourier sine transform with respect to ξ

$$\tilde{T}_s = \int_0^\infty T_s \sin(\xi\sigma)\, d\xi\,.$$

Then, by (7.2.18) and (7.2.19), one obtains for $\tilde{T}_s$ the equation

$$\sigma^4 \tilde{T}_s - \sigma^3 \theta(y) - \frac{1}{f^2}\frac{\partial \tilde{T}_s}{\partial x} = 0\,,$$

with the solution

$$\tilde{T}_s = A(y)\, e^{f^2\sigma^4 x} + \frac{1}{\sigma}\theta(y)\,, \tag{7.2.23}$$

where the function $A(y)$ must still be determined.

For solution of problems (7.2.20)–(7.2.22) and (7.2.20′)–(7.2.22′), the Fourier sine transform will again be employed. After single integrations with respect to ζ and η, respectively, one obtains from (7.2.20) and (7.2.20′) the equations

$$-f^2\sigma^2\gamma \frac{\partial \tilde{T}_W}{\partial \zeta} - \tilde{T}_W = 0\,, \tag{7.2.24}$$

$$-f^2\sigma^2\gamma \frac{\partial \tilde{T}_E}{\partial \eta} + \tilde{T}_E = 0\,. \tag{7.2.24'}$$

Equation (7.2.24′) has no non-zero solution which decays exponentially for large η. However, then the function T_E likewise vanishes identically, and, by (7.2.21′), one finds that $\tilde{T}_s(1, y, \xi) = 0$; hence the function $A(y)$ entering into (7.2.23) has been determined, and

$$\tilde{T}_s = \frac{\theta(y)}{\sigma}\{1 - \exp[-(1-x) f^2\sigma^4]\} \tag{7.2.25}$$

or, reverting to the original function,

$$T_s = \frac{2\theta(y)}{\pi}\int_0^\infty \frac{1 - \exp[-(1-x) f^2\sigma^4]}{\sigma}\sin(\sigma\xi)\, d\sigma\,. \tag{7.2.26}$$

The substitution $\sigma = \tau\xi^{1/3}(1-x)^{-1/3}$ reduces (7.2.26) to the form

$$T_s = \frac{2\theta(y)}{\pi}\int_0^\infty \frac{1 - \exp(-\chi f^2\tau^4)}{\tau}\sin(\chi\tau)\, d\tau\,, \tag{7.2.27}$$

where

$$\chi = \xi^{4/3}(1 - x)^{-1/3} . \tag{7.2.28}$$

Thus, the function T_s depends on the variables x and ξ only through the combination χ.

The function $\tilde{T}_W$ is found from (7.2.24) for the condition $T_c(0, y, \sigma) + T_W(0, y, \sigma) = 0$ [cf. (7.2.21)]. Using (7.2.25) and inverting the transform, one obtains

$$T_W = -\frac{2\theta(y)}{\pi} \int_0^\infty \frac{1 - \exp(-f^2\sigma^4)}{\sigma} \exp\left(-\frac{\zeta}{f^2\sigma^2\gamma}\right) \sin(\sigma\xi)\, d\sigma . \tag{7.2.29}$$

The integrals in (7.2.27) and (7.2.29) apparently must be evaluated numerically. However, it is not difficult to establish qualitatively the behaviour of the solution of the problem. Outside the western boundary layer, one has $T' = T_s(\chi, y)$. Therefore the lines T' = constant coincide in the x, z-plane with the lines χ = constant. By (7.2.28), these curves are given by the simple equation z^4 = constant $(1 - x)$, according to which all lines χ = constant "come out of" the point $x = 1$, $z = 0$. The "correction" function $\tau_W(\zeta, y, \xi)$ "turns upwards" these curves within the limits of a western boundary layer and forces them "into" the point $x = 0$, $z = 0$ (Fig. 7.3 on p. 202).

Thus, due to the β-effect, the pattern of the curves T' = constant is sharply asymmetric with respect to the plane $x = 1/2$, although the motion generating factor $\theta(y)$ does not at all depend on x. This phenomenon has already been encountered repeatedly.

The width of the western boundary layer is here overestimated (for the adopted values of the determining parameters, of order 200 km). However, recall again that the problem under consideration only bears a qualitative character. Besides, in essence, the parameter G has been introduced solely for linearization of the problem; for a real ocean, its estimate is very indefinite. In general, linearization of the equation of heat transfer introduces a series of artificial aspects. For example, since practically there does not occur below the thermal boundary layer a change in temperature, the vertical velocity w, by (7.2.7), will likewise be equal to zero there, and consequently also the horizontal velocity will vanish [cf., for example, (7.2.11)]. It is important to emphasize that this result follows from (7.2.7) and it does not depend on the form of the other equations. If one admits that the quantity G is not constant, then the basic state $T = T(z)$ will not satisfy the equation of heat transfer with constant exchange coefficients and the method of perturbations will then not be very sensible. All this suggests the necessity of studying non-linear models.